

# Spectral flow for Aharonov–Bohm rings generated by zero-mass lines

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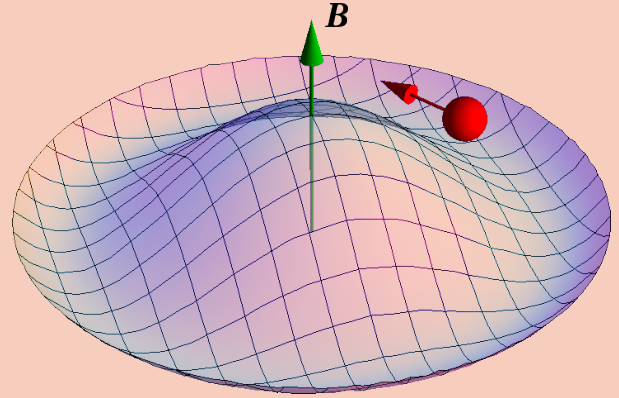
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We found that the spectrum of the linear dispersion mode localized along the zero-mass line for variable mass Dirac fermions shows the nonzero spectral flow as a function of the external magnetic field  $B$ . This happens due to the Aharonov–Bohm effect, leading to the linear dependence of every eigenvalue on  $B$ . The nonzero spectral flow allows to use the magnetic field slowly varying in time to control the energy of the linear dispersion mode.

Illustration of the Aharonov–Bohm ring generated by the ZML. The particle is shown on the mass-square landscape



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**1 Introduction** Massless Dirac fermions are charge carriers in graphene [1] and topological insulators [2], two very popular subjects in contemporary physics. Geometric and topological concepts (see, e.g., Refs. [3,4]) are of crucial importance for these systems. In particular, it was shown recently that nonzero spectral flow [5] is possible there [6,7], which means a creation of electron-hole pairs from vacuum by adiabatically changing magnetic field [7]. The works [6,7] are rather formal. Here we give a simple physical example of a situation with nonzero spectral flow. It is based on our previous consideration [8] of electron states associated with zero-mass lines (ZML). These lines appear both in graphene on hBN and other substrates and in topological insulators (e.g., in (Hg,Cd)Te/HgTe/(Hg,Cd)Te quantum wells) [8].

The peculiarities of dynamics of two-dimensional Dirac fermions localized in the vicinity of ZML were discussed in many different aspects, see [8] and references therein. Classically it is described by the Hamiltonian function

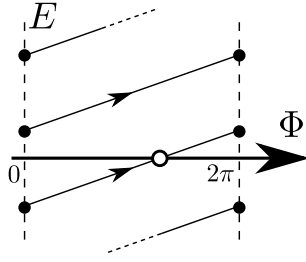
$$H(\mathbf{p}, \mathbf{r}) = \sqrt{\mathbf{p}^2 + m(\mathbf{r})^2}, \quad (1)$$

where  $\mathbf{r} = \{x, y\}$  are the coordinates,  $\mathbf{p} = \{p_x, p_y\}$  are the corresponding momenta and  $m$  is the mass. The mass square landscape plays a role of the effective potential if we choose  $H(\mathbf{p}, \mathbf{r})^2$  as an effective Hamiltonian. Thus every ZML gives rise to a one-dimensional channel, or waveguide, supporting the propagation of modes localized in its vicinity. Every mode is characterized by the transversal wave number related to the transversal energy. It turns out that there is a mode corresponding to zero transversal energy. This quasi-one-dimensional mode has a linear dispersion; i.e. its energy  $E$  of motion along the ZML is proportional to the momentum.

Let us summarize the peculiarities of the linear dispersion mode (LDM). Neglecting tunneling effects we can say that it is unidirectional, meaning that along a given ZML only the motion in a single direction is allowed. The Bohr–Sommerfeld quantization rule giving the semiclassical answer for the spectral series associated with the LDM reads

$$\oint p_\tau d\tau = 2\pi n + \phi_B + \dots, \quad (2)$$

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**Figure 1** Illustration of the spectral flow associated with the magnetic field. The spectral flow is defined as the number of crossings of the horizontal line  $E = 0$  taken with the sign of the derivative at the crossing point. In the figure spectral flow is equal to one.

where the integration should be performed along the ZML,  $\tau$  is the length of this line counted from some point,  $p_\tau$  is the longitudinal momentum,  $n$  is an integer,  $\phi_B = \pi$  for the curve without self-intersections is the Berry phase locally associated with the curvature and by ... we denoted some terms resulting from the displacement from the ZML (see [8]). For the LDM we have  $p_\tau = E$ , thus (2) takes the form  $El = 2\pi(n + 1/2) + \dots$ , where  $l$  is the length of the loop.

It is easy to modify the quantization condition for the case of non-zero external magnetic field. Indeed, the closed ZML forms an Aharonov–Bohm ring in this case. Thus, an additional magnetic flux  $\Phi = BS$  through the loop bounding area  $S$  should be added on the right:

$$El = 2\pi(n + 1/2) + \Phi + \dots \quad (3)$$

In this spectral series every eigenvalue linearly depends on the magnetic flux. If  $\Phi$  is changed (approximately) by a multiple of  $2\pi$ , then the spectral series as a whole does not change at all, except that the numbering of eigenvalues is shifted. This shift in the numbering, which can equivalently be defined as the number of crossings of the lines  $E = E(\Phi)$  with the horizontal axis  $E = 0$ , is called the *spectral flow*. Thus, the LDM gives a nontrivial physical example of nonzero spectral flow. Note that modes corresponding to nonzero transversal energy do not contribute to the spectral flow, because they always have a gap around zero. For the effective Hamiltonian corresponding to the LDM on a closed ZML, switching on a magnetic field whose magnetic flux is a multiple of  $2\pi$  is equivalent to a gauge transformation on the ZML. Of course, this transformation cannot be extended to the entire domain bounded by the ZML if the magnetic flux is nonzero. Thus, if we restrict ourselves to what happens in the vicinity of the ZML, we can adopt the point of view that the effective Hamiltonian does not change at all—we only choose a different gauge. The above-mentioned gauge transformation can be used to define a one-dimensional vector bundle on the product of the ZML by the circle, and then the spectral flow coincides with the index of the elliptic operator

that is obtained by adding the term  $\partial/\partial t$  to the effective Hamiltonian and acts on sections of this bundle,  $t$  being the coordinate on the circle.

If the magnetic field slowly varies in time, the energy levels will still obey the dependence (3). This permits controlling the energy of the LDM by means of a magnetic field: an increase in the magnetic field will lead to an increase in the energy and vice versa.

Quantum-mechanically the dynamics of the Dirac fermion in the magnetic field is described by the Hamiltonian

$$H = \boldsymbol{\sigma}(\mathbf{p} - \mathbf{A}) + \sigma_z m, \quad (4)$$

where  $\boldsymbol{\sigma} = \{\sigma_x, \sigma_y\}$  are Pauli matrices and  $\mathbf{A}$  is the vector potential. We consider a uniform magnetic field oriented along  $z$ -axis  $\mathbf{B} = \nabla \times \mathbf{A}$ .

Let us consider a bent zero line  $\gamma$  given by  $\{x, y\} = \mathbf{R}(\tau)$ , where  $\tau$  is a natural parameter, i.e.  $|\mathbf{R}'(\tau)| = 1$ . In the vicinity of this line we introduce new variables  $\tau, \xi$  by the equality  $\{x, y\} = \mathbf{R}(\tau) + \xi \mathbf{n}(\tau)$ , where  $\mathbf{n}$  is a unit normal vector on the curve at the point  $\tau$ . In the curvilinear coordinates  $\tau, \xi$  the Schrödinger equation  $H\Psi = E\Psi$  can be written as  $\hat{H}\tilde{\Psi} = E\tilde{\Psi}$ , where  $\tilde{\Psi} = \sqrt{1 - k(\tau)\xi}\Psi$ ,  $k(\tau) = -(\mathbf{R}'(\tau)\mathbf{n}'(\tau))$  is the signed curvature, and

$$\begin{aligned} \hat{H} = & -\frac{i\boldsymbol{\sigma}\mathbf{R}'(\tau)}{\sqrt{1 - \xi k(\tau)}} \frac{\partial}{\partial \tau} \frac{1}{\sqrt{1 - \xi k(\tau)}} - i\boldsymbol{\sigma}\mathbf{n}(\tau) \frac{\partial}{\partial \xi} \\ & - \frac{ik\boldsymbol{\sigma}\mathbf{n}(\tau)}{2(1 - \xi k(\tau))} - \boldsymbol{\sigma}\mathbf{A} + \sigma_z m. \end{aligned} \quad (5)$$

The Hamiltonian (5) can be simplified in the adiabatic approximation. We suppose that in the transverse direction to  $\gamma$  the wave function is localized at the scale much smaller than the radius of curvature; i.e.  $\xi k(\tau) \ll 1$ . On the other hand, it is natural to assume that the curvature does not change significantly within this scale, which implies that  $\xi k'(\tau)/k(\tau) \ll 1$ . We will consider the magnetic field  $B$  corresponding to the magnetic length  $l_0 = 1/\sqrt{B}$  comparable to the radius of the curvature  $1/k(\tau)$ . This provides a magnetic flux of the order of unity through the area bounded by the closed curve  $\gamma$ .

In the vicinity of the ZML we have  $\mathbf{A}(\tau, \xi) = \mathbf{A}(\tau, 0) + \mathbf{A}'_\xi(\tau, 0)\xi + \dots$ . Introducing the notation  $\mathbf{A}_0(\tau) = \mathbf{A}(\tau, 0)$  and using the equality  $\mathbf{A}_0(\tau) = (\mathbf{A}_0\mathbf{R}')\mathbf{R}'(\tau) + (\mathbf{A}_0\mathbf{n})\mathbf{n}(\tau)$  we find  $\hat{H} \simeq \hat{H}_0 + \hat{H}_1$ ,

$$\hat{H}_0 = \boldsymbol{\sigma}\mathbf{R}'(\tau)\hat{P}_\tau + \boldsymbol{\sigma}\mathbf{n}(\tau)\hat{P}_\xi + \sigma_z m, \quad (6)$$

$$\hat{H}_1 = \boldsymbol{\sigma}\mathbf{R}'(\tau)\xi k(\tau)\hat{p}_\tau - \frac{ik}{2}\boldsymbol{\sigma}\mathbf{n}(\tau) - \boldsymbol{\sigma}\mathbf{A}'_\xi(\tau, 0)\xi, \quad (7)$$

where  $\hat{P}_\tau = \hat{p}_\tau - (\mathbf{A}_0\mathbf{R}')$ ,  $\hat{p}_\tau = -i\partial/\partial\tau$ ,  $\hat{P}_\xi = -i\partial/\partial\xi - (\mathbf{A}_0\mathbf{n})$ . Let us choose a gauge  $\mathbf{A} = \{-By, 0, 0\}$ . Then  $\boldsymbol{\sigma}\mathbf{A}'_\xi(\tau, 0) = -\sigma_x B n_y(\tau)$ . Replacing the operator  $\hat{p}_\tau$  in (7) by the variable  $p_\tau$  we obtain symbols [9]  $H_0$  and  $H_1$ .

In the adiabatic approximation the effective dynamics along the line  $\gamma$  is described by the effective Hamiltonian  $\hat{L}_B \simeq \hat{L}_0 + \hat{L}_1$ . The symbols  $L_0$  and  $L_1$  can be found from

general considerations [10]. As usual,  $L_0$  is an eigenvalue of  $H_0$ , i.e.  $H_0\chi = L_0\chi$ . The symbol  $L_1$  is given by

$$L_1 = \langle \chi^\dagger H_1 \chi \rangle_\xi + i \left\langle \chi^\dagger \frac{\partial L_0}{\partial \tau} \frac{\partial \chi}{\partial p_\tau} \right\rangle_\xi - i \left\langle \chi^\dagger \frac{\partial H_0}{\partial p_\tau} \frac{\partial \chi}{\partial \tau} \right\rangle_\xi, \quad (8)$$

where  $\langle \cdot \rangle_\xi$  means the integration over  $\xi$ . Introducing the notation  $P_\tau = p_\tau - (\mathbf{A}_0 \mathbf{R}')$  and the function  $\chi_0$  by the equality  $\chi = e^{i(\mathbf{A}_0 \mathbf{n})\xi} \chi_0$  we reduce the calculation of  $L_0$  to the case without magnetic field [8]. It gives  $L_0 = -P_\tau$ . Similar to [8], we find from (8)

$$L_1 = G(\tau) [k(\tau)P_\tau + B] + \frac{k(\tau)}{2}, \quad (9)$$

where  $G(\tau) = 2\langle \tilde{\chi}_{01} \xi \chi_{02} \rangle_\xi$  is the displacement from ZML,  $\tilde{\chi}_{01} = (n_x - in_y)\chi_{01}$  and  $\tilde{\chi}_{01}, \chi_{02}$  are real. The explicit expression for  $G(\tau)$  is given in Ref. [8]. Thus, the effective longitudinal equation  $\hat{L}_B \psi = E\psi$  reads

$$\left( -\hat{P}_\tau + G(\tau) [k(\tau)\hat{P}_\tau + B] + \frac{k}{2} \right) \psi = E\psi, \quad (10)$$

Equation (10) can be solved in the semiclassical approximation. The solution is

$$\psi = \exp \left( -iE\tau + i \int \mathbf{A}_0 d\mathbf{R} \right) \times \exp \left( i \int G(\tau) [B - k(\tau)E] d\tau + \frac{i\phi}{2} \right). \quad (11)$$

where  $\phi_B = \phi/2$  is the Berry phase locally associated with the curvature [8],  $d\phi = k(\tau)d\tau$ . Here  $\phi$  is none other than the angle between the tangent to the curve  $\gamma$  and the  $x$ -axis, so that it gets an increment of  $2\pi$  when going around  $\gamma$ . Using the answer (11) it is easy to find the quantization condition for the case in which  $\gamma$  is a closed curve:

$$E \left( l + \oint G(\tau)k(\tau)d\tau \right) = 2\pi \left( n + \frac{1}{2} \right) + B \left( S + \oint G(\tau)d\tau \right), \quad (12)$$

where the terms given by the circular integrals describe the small corrections to the length  $l$  of the contour  $\gamma$  and the area  $S$  bounded by this contour owing to the displacement of the wave function from  $\gamma$ . Thus, the increase of the total flux by flux quantum corresponds to the replacement  $n \rightarrow n + 1$ . As a result, the spectral flow is equal to one, as illustrated in Fig. 1.

Below we give a formal proof of this intuitively clear statement. Assume that the magnetic flux

$$\Phi = B \left( S + \oint G(\tau)d\tau \right)$$

is an integer multiple of  $2\pi$ ,  $\Phi = 2\pi m$ . Then it follows from the quantization condition (12) that the spectrum of the effective Hamiltonian  $\hat{L}_B$  is the same as for  $B = 0$ . In other words,  $\hat{L}_B$  and  $\hat{L} = \hat{L}_B|_{B=0}$  are *isospectral*. In fact,  $\hat{L}_B$  can be obtained from  $\hat{L}$  by a gauge transformation,  $\hat{L}_B = U^{-1}\hat{L}U$ , where

$$U = \exp \left( -i \int \mathbf{A}_0 d\mathbf{R} - B \int G(\tau) d\tau \right).$$

Consider the family of operators

$$\hat{L}(t) = (1-t)\hat{L} + t\hat{L}_B, \quad 0 \leq t \leq 1.$$

The spectral flow arising when the magnetic field  $B$  is switched on is just the spectral flow  $\text{sf } \hat{L}(t)$  of this family. The latter can be expressed as the index of some elliptic operator. Namely, consider the first-order differential operator  $\partial/\partial t + \hat{L}(t)$  on the product  $\gamma \times [0, 1]$ . We glue the endpoints of the interval  $[0, 1]$  together, so that the interval becomes the circle  $S^1$  and this product becomes the torus  $\gamma \times S^1$ . The operator  $\partial/\partial t + \hat{L}(t)$  becomes an operator with continuous coefficients on this product if we use the transformation  $U$  to glue together the values of the functions on  $\gamma \times \{0\}$   $\gamma \times \{1\}$ . One can readily verify that this operator on  $\gamma \times S^1$  is elliptic, and by [5]

$$\text{sf } \hat{L}(t) = \text{ind} \left( \frac{\partial}{\partial t} + \hat{L}(t) \right),$$

where ‘ind’ denotes the index of the operator. Thus, the spectral flow is equal to the number of zero-modes of the operator  $\partial/\partial t + \hat{L}(t)$  minus that of the adjoint operator, that is, one.

To conclude, we have shown that a closed zero-mass line naturally generates the Aharonov–Bohm ring for the Dirac fermions if the external magnetic field is applied. For the linear dispersion mode the Aharonov–Bohm effect shifts the entire (sub)spectrum, leading to nonzero spectral flow. This effect can be used to control the energy of the linear dispersion mode by means of the magnetic field slowly changing in time. The spectral flow can be computed in topological terms as the index of an elliptic operator on the two-dimensional torus.

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